THE DETERMINANT FORMULA IN THE INVERSE SCATTERING PROCEDURE FOR STEPLIKE POTENTIALS

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A

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Abstract

The inverse scattering procedure associated with the one-dimensional Schrödinger operator $H = -\partial_x^2 + q(x)$ consists of reconstructing $q(x)$, $x \in \mathbb{R}$, based on solutions to the eigenvalue problem for H. One method in particular recovers the potential in terms of the Fredholm determinant of the so-called Marchenko operator when the potential decays sufficiently fast on the line. In this paper, we derive a similar formula but for steplike potentials, i.e. potentials with sufficient decay on one half line but no decay on the other half line.

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Acknowledgments

This particular paper is a joint work with Dr. Alexei Rybkin, my advisor. I would like to thank him first and foremost for providing expertise, encouragement, and guidance throughout my degree. He spent countless hours shaping my knowledge, giving me many opportunities to conduct research and to assist him in his. In addition, I feel privileged to have been the recipient of his kindness and friendship. My thanks also go to my other committee members: Dr. Jill Faudree for her encouragements and Dr. David Maxwell for providing an excellent foundation in the core subjects. I have enjoyed the classes taken from them and the qualities they displayed therein made them natural candidates to be on my committee. Finally, this degree would not have been possible without the patience and support of my family: my husband Derek and my wonderful children Talina and Louis. All three took on more than their fair share around the house to allow me to pursue my interest in math and for that I am greatly indebted to them.

Dedication

This thesis is dedicated to J.-J. de Botton who recently retired after close to 30 years of teaching math at the middle school I attended. He was my math teacher during two of the most formative years of my education and had a tremendous impact on shaping my future.

During class, Mr. de Botton always addressed us very formally, using only our last names yet his teaching style was all about keeping our minds sharp, alert through fun, quick games. New concepts were explained in an engaging way – one time we spent twenty minutes on practicing on an imaginary function called "schmoll", all in anticipation of learning about something called square root. He saw opportunities to practice math at every corner. I recall a field trip to the Versailles castle where he had the whole bus recite the Pythagorean theorem. Mr. de Botton also created lasting impressions by having exciting previews of what we would learn beyond his class. For example, he would say something like: "Remember those points on the graph? Next year you will see them move, it's called a vector!" I could not wait for summer to be over so that I could learn about all these unknown topics.

I went into Mr. de Botton's class an average math student with a primary interest in French and came out with not only much stronger skills but also a lifelong interest in pursuing math related subjects. Despite a challenging learning environment, Mr. de Botton saw in me – and, I believe, in many others – a potential for a career in mathematics and was instrumental in one particular instance in securing for me a continuing education. At first resisting the idea (while he was signing me up for Greek, I was signing up for Latin), I came to quickly embrace studies in science and a career in teaching where I could draw inspiration from his passion and dedication. This lasting influence remained unbeknownst to him until recently. When I was about to embark on this degree, I showed up unannounced at his doorstep after twenty years with no contact. He came to the gate, extended his hand and called out my last name.

Beyond the formality of this greeting was the enduring interest Mr. de Botton displays in the continuing success of his students. Similarly, while also formal, this paper is but a token of my gratitude towards an exceptional teacher and person.

Chapter 1
Introduction

One of the main objects used in studying the physical world is related to the *scattering* phenomenon, that is the interaction between waves or particles and a medium or target at which they are aimed [10]. Namely, an incident wave or particle, when confronted with a *potential* (discontinuity in the medium or object), deviates from its original path, and at large distances, one can record the part of the normal wave which was transmitted and the part reflected because of the potential. In the *inverse scattering* problem, one uses the *scattering data* collected to reconstruct the potential. The use of *reflection coefficients* obtained after emitting particular waves – e.g. acoustic, seismic, X-ray – is key to recovering the properties of distant objects in the fields of echolocation, geophysical surveys, and medical imaging for example.

In our present mathematical context, the potential $q(x)$ vary along the real line. This is referred to as *potential scattering*. The role of the wave is played by generalized solutions of the eigenvalue problem for the one-dimensional Schrödinger operator:

$$H = -\partial_x^2 + q(x).$$

on the Hilbert space $L^2(\mathbb{R})$. This example is canonical and has direct applications to atomic nuclear physics and acoustic wave scattering for example. Methods to recover the potential, i.e. *inverse scattering procedures* for this problem, have been developed since the 1950s, particularly in the case of so-called *Faddeev* potentials (see e.g. [1, 5][1]), i.e. such that $\int_{\mathbb{R}} (1+|x|)\,|q(x)|\,dx$ is finite. In the *direct scattering* problem, one analyzes solutions to $H\Psi(x,k) = k^2\Psi(x,k)$ where k^2 is referred to as energy. One may find *bound states* corresponding to $\{-\varkappa_n^2\}_{n=1}^N$ among the negative energies in addition to defining a reflection coefficient $R(k)$ for positive energies. Then the scattering data comprising not only the reflection coefficient but also bound states and associated *norming constants* $\{c_n\}_{n=1}^N$ determine uniquely the potential. In the Marchenko inverse scattering procedure, the potential

[1]Some aspects of the theory in [5] actually required the stronger condition

$$\int_{\mathbb{R}} \left(1+x^2\right) |q(x)|\,dx < \infty.$$

is then recovered through the following formula, sometimes referred to as Bargmann or Dyson (see e.g. [7]):

$$q(x) = -2\partial_x^2 \log\det(1 + \mathbb{M}_x)$$

where $\mathbb{M}_x$ is called the Marchenko operator defined in terms of the scattering data S.

In the present paper we study the inverse scattering for a broader class of potentials, namely we assume that $q(x)$ is Faddeev on $(0, \infty)$ but only locally integrable on $(-\infty, 0)$ with

$$\sup_{x \leq 0} \int_{x-1}^{x} |q(s)|^2 \, ds < \infty.$$

We refer to such potentials as *steplike* since $q(x)$ decays sufficiently fast on $(0, \infty)$ to apply classical results but need not decay at $-\infty$.

Our study is motivated by results from Gesztesy-Simon [9] which showed that the inverse problem

$$\{R(k), q(x)\}_{k \in \mathbb{R}, x \geq 0} \quad \longrightarrow \quad q(x)\ ,\ x < 0$$

is well-posed for essentially arbitrary potentials. His approach uses properties of so-called Weyl solutions of the half-line Schrödinger operators but follows the Gelfand-Levitan inverse procedure which is based on the spectral function. Recently, Rybkin [12] used similar arguments to adjust the Marchenko inverse scattering procedure to this setting for a potential vanishing on the right hand side.

We continue along this approach to solve the inverse problem for a *partially known* potential, i.e. the potential is a priori known on $[0, \infty)$. Our main result is that the potential can be uniquely recovered through the determinant formula

$$q(x) = -2\partial_x^2 \log\det\left(1 + (1 + \mathbb{M}_x^+)^{-1}\mathbb{G}_x\right), \quad x < 0,$$

where $\mathbb{M}_x^+$ is the Marchenko operator associated to $q(x)|_{\mathbb{R}_+}$ and $\mathbb{G}_x$ is a certain trace class integral operator expressed in terms of the difference of two suitably defined reflection coefficients. We show that the determinant above is well-defined as a classical Fredholm determinant. This implies that discretization in possible applications will be stable.

The paper is organized as follows. Chapter 2 introduces the direct scattering problem and all scattering quantities present in our problem. Chapter 3 details relevant properties of the Titchmarsh-Weyl m-function which are then applied in our context in Chapter 4. Chapter 5 reviews the classical Marchenko inverse scattering procedure as well as introduce an important property of certain integral operators. Chapter 6 contains the main result and conclusions.

1.1 Notation

For the reader's convenience, we introduce here all relevant notation used in the paper. We use $\pm$ to indicate two separate statements. We adhere to standard terminology from analysis, namely $\mathbb{R}_\pm := [0, \pm\infty)$, $\mathbb{C}$ is the complex plane, $\bar{z}$ denotes the complex conjugate of z,

$$\mathbb{C}^+ = \{z \in \mathbb{C} \ : \ \operatorname{Im} z > 0\} \quad , \quad i\mathbb{R}_+ = \{z \in \mathbb{C} \ : \ z = iy \ , \ y \in \mathbb{R}_+\} \, .$$

In the upper half plane,

$$\mathbb{R} + ih = \{z \in \mathbb{C} \ : \ z = x + ih \ , \ x \in \mathbb{R}\}$$

is the real line shifted h units up.

The Wronskian in x of two differentiable functions u, v is

$$W(u,v) := u\partial_x v - v\partial_x u,$$

where ∂_x is the partial derivative in x.

$\|\cdot\|_X$ stands for a norm in a Banach (Hilbert) space X. We use (f is measurable, $S \subseteq \mathbb{R}$ and S will typically be $\mathbb{R}$ or $\mathbb{R}_\pm$):

- the usual Lebesgue spaces

$$L^p(S) := \left\{ f : \|f\|_{L^p(S)} := \left(\int_S |f(x)|^p \, dx \right)^{1/p} < \infty \right\} \quad , \quad 1 \le p < \infty$$

$$L^\infty(S) := \left\{ f : \|f\|_{L^\infty(S)} := \operatorname*{ess\,sup}_{x \in S} |f(x)| < \infty \right\},$$

$$L^p_{loc}(S) := \{\cap L^p(\Delta) : \Delta \subset S, \ \Delta \text{ compact } \} \, .$$

- the Faddeev class

$$L_1^1(S) = \left\{ f : \|f\|_{L_1^1(S)} := \int_S (1+|x|)\,|f(x)|\,dx < \infty \right\}.$$

- the Birman-Solomjak spaces ($1 \le p < \infty$)

$$\ell^\infty(L^p(\mathbb{R}_\pm)) := \left\{ f : \|f\|_{\ell^\infty(L^p(\mathbb{R}_\pm))} := \sup_{x\in\mathbb{R}_\pm} \left(\int_x^{x\pm 1} |f(x)|^p\,dx \right)^{1/p} < \infty \right\}.$$

Next, $\mathfrak{S}_2$ denotes the Hilbert-Schmidt class of linear operators A:

$$\mathfrak{S}_2 = \left\{ A : \|A\|_{\mathfrak{S}_2} := [\operatorname{tr}(A^*A)]^{1/2} < \infty \right\}$$

while $\mathfrak{S}_1$ is the trace class

$$\mathfrak{S}_1 = \left\{ A : \|A\|_{\mathfrak{S}_1} := \operatorname{tr}\left[(A^*A)^{1/2} \right] < \infty \right\}.$$

$\operatorname{Spec}(A)$ stands for the spectrum of an operator A and if it is selfadjoint, $\operatorname{Spec}_{ac}(A)$, $\operatorname{Spec}_d(A)$ denote respectively the absolutely continuous and discrete components of the spectrum of A.

The following portion of notation will be used extensively in reference to the potential q and quantities associated with it. If $\chi_S(x)$ is the characteristic function of a set S, i.e. $\chi_S(x) = 1, x \in S$ and 0 otherwise, then we define:

$$q_+(x) := q(x)\chi_{\mathbb{R}_+}(x)\ ,\ q_-(x) = q(x)\chi_{\mathbb{R}_-}(x)\ ,\ \tilde{q}(x) = q(x)\chi_{[-a,a]}(x) \text{ for some } a > 0.$$

We also write

$$\delta q := q - \tilde{q}$$

and when the cutoff approximation is taken to infinity, i.e. $a \to \infty$, we write $\delta q \to 0$.

Any quantity X of arbitrary nature (functions, operators, etc.) related to $\tilde{q}$ will be denoted $\tilde{X}$ and we define

$$\delta X := X - \widetilde{X}.$$

Chapter 2

The direct scattering problem for the Schrödinger equation

In this chapter, we study the direct scattering problem, i.e. characterizing solutions $\Psi(x,k)$ to the Schrödinger equation

$$-\partial_x^2\Psi(x,k) + q(x)\Psi(x,k) = k^2\Psi(x,k) \tag{2.1}$$

with the spectral parameter k^2 generally called energy, and k called momentum. In short notation, we write $H\Psi = k^2\Psi$ where $H = -\partial_x^2 + q(x)$ is the one-dimensional Schrödinger operator. We assume that $q(x)$ is real-valued and so one easily verifies that H is self-adjoint when acting in the Hilbert space $L^2(\mathbb{R})$. Therefore, its spectrum is a subset of $\mathbb{R}$. Our arguments will call for deforming a contour along $k \in \mathbb{R}$ to the upper half plane so an important consideration is the nature of the spectrum and in particular if it is bounded below as well as analytic properties of solutions in $\mathbb{C}^+$.

In the case of fast decaying potentials, scattering can be seen as a perturbation [7] of the free Schrödinger operator $H_0 = -\partial_x^2$ with wave solutions $e^{\pm ikx}$ for $k \in \mathbb{R}$. Such waves incident from $\pm\infty$ are indeed propagating unperturbed through a potential $q(x) = 0$. The existence of solutions with asymptotics $e^{\pm ikx}$ at the infinities, called *Jost solutions*, are at the core of the scattering problem for Faddeev potentials.

In the more general context, the Titchmarsch-Weyl theory applies which does not assume decay of the potential at either infinity. The case of the full-line Schrödinger equation is reduced to considering two half-lines $\mathbb{R}_\pm$, so we will split the potential accordingly:

$$q(x) = q_+(x) + q_-(x).$$

We recall the following definition relevant to Weyl theory [17, 18].

Definition 2.1. *Let $q(x)$ be locally integrable and consider solutions to $(H-\lambda)y = 0$ on $\mathbb{R}$. Then, for every value of λ other than real values,*

- *either there is exactly one (up to a multiplicative constant) solution belonging to $L^2(\mathbb{R}_\pm)$; this is referred to as the* limit point case *at $\pm\infty$,*
- *or all solutions belong to $L^2(\mathbb{R}_\pm)$ and this is called the* limit circle case *at $\pm\infty$.*

Note that any Faddeev potential is limit point at both infinities [14]. In the trivial case $q(x) = 0$, then we have for example that e^{ikx} is the only solution (up to an additive multiple) of (2.1) in $L^2(\mathbb{R}_\pm)$ for $\pm \operatorname{Im} k > 0$.

Throughout this chapter we assume the following.

Hypothesis 2.2. *The potential q is real, locally integrable and such that*

(1) q is limit point case at $-\infty$,

(2) q_+ is Faddeev class and hence (2.1) *has Jost solutions at $+\infty$.*

2.1 Description of the spectrum of the Schrödinger operator

Under Hypothesis 2.2, H is selfadjoint and its spectrum is contained in $\mathbb{R}$. Further characterization is difficult unless more is known about the potential.

By Hypothesis 2.2 condition (2), the spectrum of the Schrödinger operator associated with the potential q_+ exhibits all properties known in classical scattering theory, namely (see e.g. [1, 6]):

- the absolutely continuous part of the spectrum covers the whole right half line and is of multiplicity 2; that is

$$\operatorname{Spec}_{ac}(-\partial_x^2 + q_+) = \mathbb{R}_+$$

and there exist bounded, not square integrable solutions to

$$-\partial_x^2 u(x,k) + q_+(x)u(x,k) = k^2 u(x,k)$$

for all $k \in \mathbb{R}$.

- the discrete part of the spectrum is finite, simple and contained in the left half line; i.e.

$$\operatorname{Spec}_d(-\partial_x^2 + q_+) = \{-\varkappa_n^2\}_{n=1}^N \tag{2.2}$$

and the associated eigenfunctions are referred to as bound states[1].

This is illustrated in Figure 2.1.

[1]This is in direct connection with quantum physics terminology.

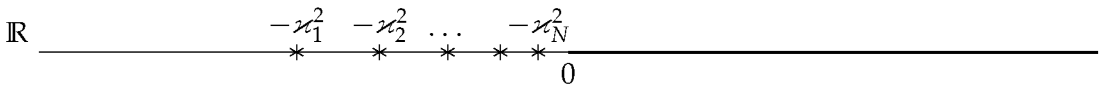

Figure 2.1. Spectrum of the Schrödinger operator $-\partial_x^2 + q_+(x)$

But the condition locally integrable and limit point at $-\infty$ covers a wide range of options for $\mathrm{Spec}(-\partial_x^2 + q_-)$ and $\mathrm{Spec}(H)$. For example, if $q_- \in L^1(\mathbb{R})$ but not $L^1_1(\mathbb{R})$, then eigenvalues may accumulate at zero and their number becomes infinite [1, 8]. In the case of the nondecaying potential $q_-(x) = -c^2\chi_{\mathbb{R}_-}(x)$ where c is a real constant, then the continuous spectrum of $-\partial_x^2 + q_-(x)$ extends in the negative axis with $\mathrm{Spec}_{ac}(-\partial_x^2 + q_-(x)) = [-c^2, \infty)$. Finally, if $q_-(x) = -x^2\chi_{\mathbb{R}_-}(x)$, then the spectrum covers all of $\mathbb{R}$ [18]. We will not consider this last case since we are interested in having $\mathrm{Spec}(H)$ bounded below. Based on the work by Gesztesy-Simon [9] though, we need to ensure $\mathrm{Spec}_{ac}(H)$ is not empty. This is already guaranteed by q_+ being Faddeev.

2.2 Weyl solutions of the Schrödinger equation

By Hypothesis 2.2 condition (1), the equation (2.1) has a unique, up to a multiplicative constant, solution Ψ_-, called *Weyl solution*, such that $\Psi_-(x,k) \in L^2(\mathbb{R}_-)$ for any $k^2 \in \mathbb{C}^+$. Condition (2) implies that q is limit point case at $+\infty$ and the Weyl solution $\Psi_+(x,k)$ can be taken to have the asymptotic behavior: (for any real k)

$$\Psi_+(x,k) = e^{ikx} + o(1) \quad , \quad x \to \infty.$$

The Weyl solution Ψ_+ coincides in this case with the Jost solution. Furthermore, $\Psi_+, \overline{\Psi}_+$ are both solutions to (2.1) and linearly independent for almost every (a.e.) real k with constant Wronskian

$$W := W\left(\overline{\Psi_+(x,k)}, \Psi_+(x,k)\right) = 2ik. \tag{2.3}$$

Hence they form a basis of solutions for (2.1) for a.e. real k and in particular:

$$C(k)\Psi_-(x,k) = \overline{\Psi_+(x,k)} + R(k)\Psi_+(x,k) \tag{2.4}$$

with some $C(k), R(k)$. We call R the *reflection coefficient* from the right incident.

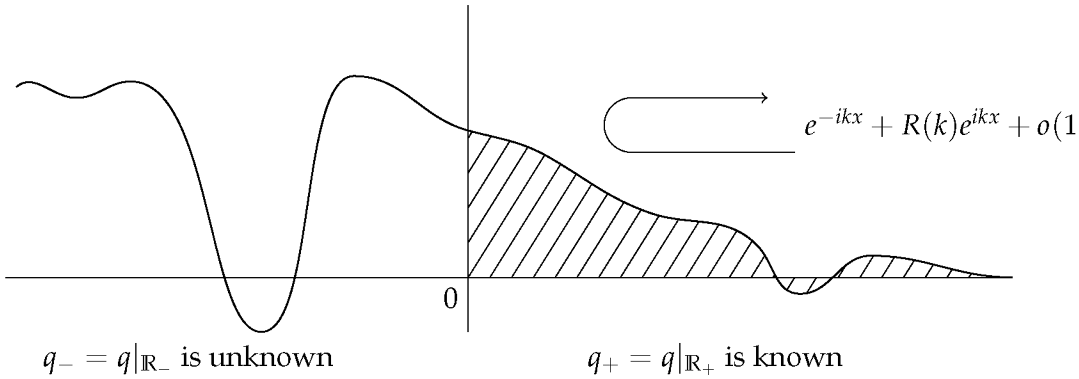

Figure 2.2. Scattering channels for $q = q_- + q_+$

Under our hypothesis, neither C nor R can be analytically continued into the upper half plane. Figure 2.2 illustrates a potential from our hypothesis and the asymptotic behavior of $C\Psi_-$ at $+\infty$.

2.3 Scattering solutions for the potentials q_+, q_-

We now consider separately scattering solutions corresponding to $q_\pm$. We are interested in particular solutions which are a linear combination of $e^{\pm ikx}$ on the vanishing half line of the potential and correspond to a multiple of the relevant Weyl solution – as described in the previous section – on the other half line.

Indeed, first by our hypothesis at $-\infty$, there is a solution $\varphi_-(x,k)$ to

$$-\partial_x^2 u + q_- u = k^2 u$$

of the form: ($k \in \mathbb{R}$)

$$\varphi_-(x,k) = \begin{cases} D(k)\Psi_-(x,k) & , \quad x < 0 \\ e^{-ikx} + R_-(k)e^{ikx} & , \quad x \geq 0 \end{cases} \tag{2.5}$$

with some $D(k), R_-(k)$ (see Figure 2.3). We again refer to R_- as a reflection coefficient from the right incident but associated with q_-.

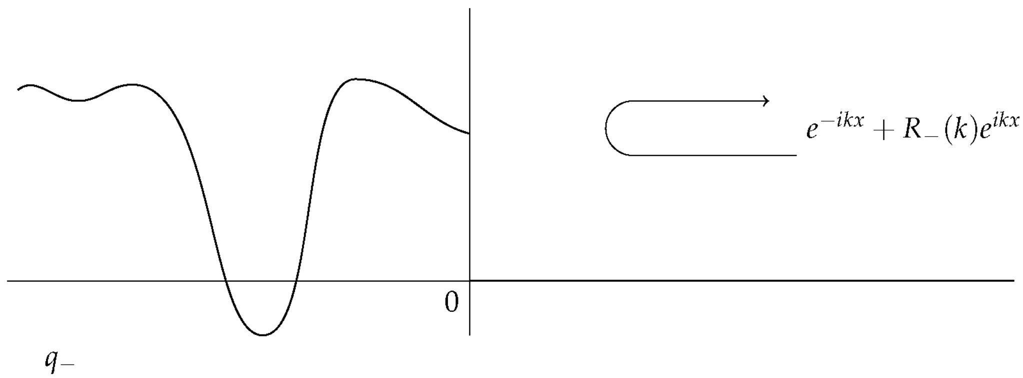

Figure 2.3. Scattering channels for q_-

For q_+, there exist particular *Jost solutions* $\varphi_{\ell,+}$ and $\varphi_{r,+}$ to

$$-\partial_x^2 u + q_+ u = k^2 u$$

such that: ($k \in \mathbb{R}$)

$$T_+(k)\varphi_{\ell,+}(x,k) = \begin{cases} e^{ikx} + L_+(k)e^{-ikx} & , \quad x < 0 \\ T_+(k)\Psi_+(x,k) & , \quad x \geq 0 \end{cases} \tag{2.6}$$

$$T_+(k)\varphi_{r,+}(x,k) = \begin{cases} T_+(k)e^{-ikx} & , \quad x < 0 \\ \overline{\Psi_+(x,k)} + R_+(k)\Psi_+(x,k) & , \quad x \geq 0 \end{cases} \tag{2.7}$$

where T_+ is called the *transmission coefficient*, and L_+, R_+ are the reflection coefficients from respectively the left or right incident associated with q_+. Because q_+ is Faddeev, it is well-known [5] that the transmission coefficient T_+ can be analytically continued in the upper half plane. Furthermore, T_+ has only a finite number of simple poles $\{i\varkappa_n\}_{n=1}^N$ corresponding to (2.2).

Since, in addition, q_+ is supported on the right half line, the reflection coefficient L_+ can also be analytically extended to the upper half plane and shares the same poles as T_+. However, in general, the reflection coefficient R_+ can not be extended off the real axis (see e.g. [1]). The asymptotic behavior of these solutions is illustrated in Figure 2.4.

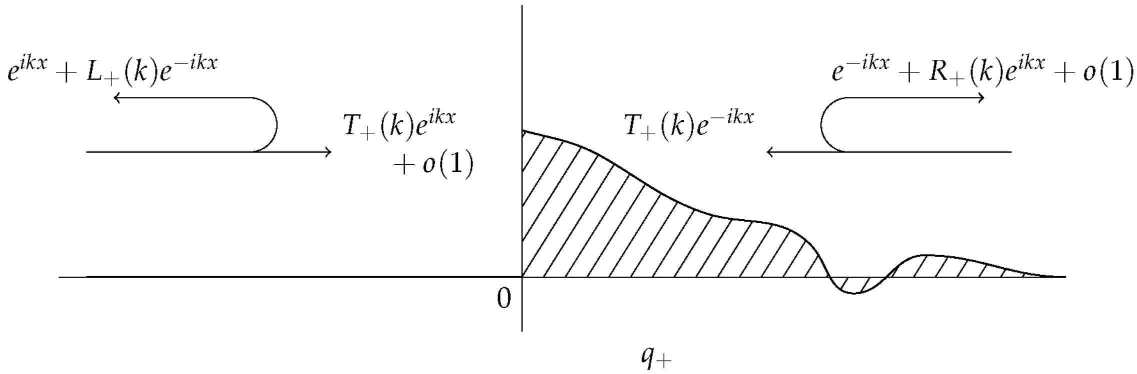

Figure 2.4. Scattering channels for q_+

Note that using (2.4)-(2.7), all right reflection and transmission coefficients can be expressed in terms of Wronskians of particular solutions of the Schrödinger equation on the line or half lines. Of particular interest, we have for a.e. real k:

$$W = 2ik = W(\Psi_+(x,k), C(k)\Psi_-(x,k))$$

$$R(k) = \frac{W\left(\overline{\Psi_+(x,k)}, C(k)\Psi_-(x,k)\right)}{W} \tag{2.8}$$

$$R_-(k) = \left.\frac{W\left(e^{-ikx}, \varphi_-(x,k)\right)}{W}\right|_{x\geq 0} \tag{2.9}$$

$$R_+(k) = \left.\frac{W\left(\overline{\Psi_+(x,k)}, T_+(k)\varphi_{r,+}(x,k)\right)}{W}\right|_{x\geq 0} \tag{2.10}$$

$$\frac{1}{T_+(k)} = \frac{W\left(\varphi_{r,+}(x,k), \varphi_{\ell,+}(x,k)\right)}{W} \tag{2.11}$$

The analytic continuation of the transmission coefficient T_+ in the upper half plane is clearly recognizable from (2.11) once one shows the analyticity of $\varphi_{\ell,+}$ and $\varphi_{r,+}$ in $\mathbb{C}^+$. This approach can be found in Deift and Trubowitz's important paper [5].

However, one can not readily derive such results from the expressions (2.8)-(2.10) for the reflection coefficients R, R_-, R_+. We refer though the reader to the next chapter where the logarithmic derivatives of the Weyl solutions, known as the *Titchmarsh-Weyl m-functions*, are analyzed.

As a final note, we remark that any truncation $\tilde{q} = q|_{[-a,a]}$ is compactly supported which implies that $\widetilde{R}, \widetilde{R}_+$ can be analytically continued into $\mathbb{C}^+$ [5] except at a finite number of poles located on $i\mathbb{R}_+$ such that their squares correspond respectively to the discrete spectrum of $-\partial_x^2 + \tilde{q}$ and $-\partial_x^2 + \tilde{q}_+$.

Chapter 3

The Titchmarsh-Weyl m-function

In this chapter, we review properties of the Titchmarsh-Weyl m-function. This function will be a central object in redefining scattering quantities in the next section. We will have to impose some additional conditions on the potential q. Most of the material presented here appeared in [12].

In the Titchmarsh-Weyl theory [11, 18], a Weyl solution of $(H-\lambda)u = 0$ on the half line is originally defined as a linear combination of two solutions $\theta(x,\lambda), \phi(x,\lambda)$ with the following prescribed boundary conditions at $x = 0$:

$$\theta(0,\lambda) = 1, \quad \partial_x \theta(0,\lambda) = 0,$$
$$\phi(0,\lambda) = 0, \quad \partial_x \phi(0,\lambda) = 1.$$

Then

$$W(x,k) = \theta(x,k) + m(k)\phi(x,k)$$

is the Weyl solution which satisfies $W(0,k) = 1$ and $\partial_x W(0,k) = m(k)$. The function $m(k)$ is called the Titchmarsh-Weyl m-function. We will use an alternate but equivalent definition [11].

Definition 3.1. *The Titchmarsh-Weyl m-function is defined by:*

$$m_\pm(k^2) = \pm \frac{\partial_x \Psi_\pm(x,k)}{\Psi_\pm(x,k)}\bigg|_{x=\pm 0}.$$

Some of the important properties of the m-function are (see e.g. [11, 12, 18]):

- $m_\pm$ is analytic for all $k^2 \in \mathbb{C}^+$ and has the Herglotz property, i.e. $m_\pm : \mathbb{C}^+ \to \mathbb{C}^+$.
- symmetry $m_\pm(\bar{z}) = \overline{m_\pm(z)}$.
- the singularities of $m_\pm$ correspond to the spectrum of the half line Dirichlet Schrödinger operator, i.e. $-\partial_x^2 + q$ on $\mathbb{R}_\pm$ with solutions satisfying $u(\pm 0) = 0$.
- the Borg-Marchenko uniqueness theorem: $m_1 = m_2 \;\Rightarrow\; q_1 = q_2$.

The following representation of the Titchmarsh-Weyl m-function $m_\pm$ will be useful.

Proposition 3.2. *Let q be a real-valued function on $\mathbb{R}$ such that $q \in \ell^\infty(L^2(\mathbb{R}_-)) \cap L^1(\mathbb{R}_+)$. Let $\gamma = \max(\gamma_-, \gamma_+)$ where*

$$\gamma_- = \max\left(\sqrt{2\,\|q_-\|_{\ell^\infty(L^2(\mathbb{R}_-))}}, e\,\|q_-\|_{\ell^\infty(L^2(\mathbb{R}_-))}\right),$$
$$\gamma_+ = \frac{\|q_+\|_{L^1(\mathbb{R}_+)}}{2}.$$

Then for $k = \alpha + ih$, $h > \gamma$,

$$m_\pm(k^2) = ik \mp \int_0^{\pm\infty} e^{\pm 2ikx} A_\pm(x)dx \tag{3.1}$$

with some real function $A_\pm(x)$, called the A-amplitude. The integral in (3.1) *is absolutely convergent and the A-amplitude has the following properties.*

(1) $A_\pm - q_\pm$ is continuous on $\mathbb{R}_\pm$ and for $\pm x \geq 0$:

$$|A_\pm(x) - q_\pm(x)| \leq \left(\pm\int_0^x |q_\pm(s)|\,ds\right)^2 e^{\pm 2\gamma x} \tag{3.2}$$

(2) If $q_1, q_2 \in \ell^\infty(L^2(\mathbb{R}_-)) \cap L^1(\mathbb{R}_+)$ then

$$q_1(x) = q_2(x) \text{ on } [0, \pm a] \quad \Rightarrow \quad A_1(x) = A_2(x) \text{ on } [0, \pm a]. \tag{3.3}$$

(3) For any $h > \gamma$,

$$\left\| e^{\mp 2hx} A_\pm(x) \right\|_{L^1(\mathbb{R}_\pm)} \leq C(h, q_\pm) < \infty$$

and $C(h, q_\pm)$ is a nonincreasing function of h.

(4) For any $h > \gamma$,

$$e^{2hx} A_-(x) \in L^2(\mathbb{R}_-).$$

Proof. The representation (3.1) was found in [11] for potentials q in the Faddeev class and in [14] for more general q's. Note first that such potential is limit point at $\pm\infty$ [14] so $m_\pm$ are well-defined. Properties (3.2)-(3.3) were derived for $q_+ \in L^1(\mathbb{R}_+)$ in [14] then for

$q_+ \in \ell^\infty(L^1(\mathbb{R}_+))$ in [3] but since

$$m_-(q_-(x), k^2) = m_+(q_-(-x), k^2)$$

and $\ell^\infty(L^2(\mathbb{R}_-)) \subset \ell^\infty(L^1(\mathbb{R}_-))$, we have adjusted the results accordingly. So only (3)-(4) require a proof. We will consider A_- and $p = 1, 2$. Using Minkowski's inequality, one needs only to show $e^{2hx}q_-(x)$ and $e^{2hx}(A_-(x) - q_-(x))$ are in $L^p(\mathbb{R}_-)$. Dropping the subscripts, we have

$$\begin{aligned}
\int_{-\infty}^{0} \left| e^{2hx} q(x) \right|^p dx &= \int_{-\infty}^{0} e^{2hpx} \left| q(x) \right|^p dx \\
&= \sum_{n=0}^{\infty} \int_{-n-1}^{-n} e^{2hpx} \left| q(x) \right|^p dx \\
&\leq \sum_{n=0}^{\infty} e^{-2hpn} \int_{-n-1}^{-n} \left| q(x) \right|^p dx \\
&\leq \sum_{n=0}^{\infty} e^{-2hpn} \left\| q(x) \right\|^p_{\ell^\infty(L^p(\mathbb{R}_-))} \\
&= \frac{1}{1 - e^{-2hp}} \left\| q(x) \right\|^p_{\ell^\infty(L^p(\mathbb{R}_-))}.
\end{aligned}$$

For the next term, we will make use of the following: $(x \leq 0)$

$$\begin{aligned}
\int_{x}^{0} |q(s)|\, ds \leq \sum_{n=1}^{-\lfloor x \rfloor} \int_{-n}^{-n+1} |q(s)|\, ds &\leq \sum_{n=1}^{-\lfloor x \rfloor} \|q\|_{\ell^\infty(L^1(\mathbb{R}_-))} \\
&\leq (1 - x) \|q\|_{\ell^\infty(L^1(\mathbb{R}_-))} \\
&\leq (1 - x) \|q\|_{\ell^\infty(L^p(\mathbb{R}_-))}
\end{aligned}$$

where the last inequality is a direct consequence of Hölder's inequality. Hence,

$$\begin{aligned}
\int_{-\infty}^{0} \left| e^{2hx}(A(x) - q(x)) \right|^p dx &= \int_{-\infty}^{0} e^{2hpx} \left| A(x) - q(x) \right|^p dx \\
&\leq \int_{-\infty}^{0} e^{2p(h-\gamma)x} \left(\int_{x}^{0} |q(s)|\, ds \right)^{2p} dx \\
&\leq \int_{-\infty}^{0} e^{2p(h-\gamma)x} (1 - x)^{2p} \|q\|^{2p}_{\ell^\infty(L^p(\mathbb{R}_-))}\, dx.
\end{aligned}$$

One readily verifies that for any $m = 0, 1, 2, \ldots$ and $b > 0$

$$\int_{-\infty}^{0} (1-x)^m e^{bx} dx = \frac{m!}{b^{m+1}} \sum_{k=0}^{m} \frac{b^k}{k!}.$$

Therefore, we have

$$\int_{-\infty}^{0} \left| e^{2hx} (A(x) - q(x)) \right|^p dx \leq \frac{2p!}{[2p(h-\gamma)]^{2p+1}} \sum_{j=0}^{2p} \frac{[2p(h-\gamma)]^j}{j!} \|q\|_{\ell^\infty(L^p(\mathbb{R}_-))}^{2p}.$$

So (3)-(4) are verified with

$$C(h, q_-) = \frac{1}{1-e^{-2h}} \|q_-(x)\|_{\ell^\infty(L^1(\mathbb{R}^-))} + \frac{1}{4\,(h-\gamma)^3} \sum_{j=0}^{2} \frac{[2(h-\gamma)]^j}{j!} \|q_-\|_{\ell^\infty(L^1(\mathbb{R}_-))}^2.$$

Similarly for A_+, we can take in (3)

$$C(h, q_+) = \|q_+\|_{L^1(\mathbb{R}_+)} + \frac{1}{h-\gamma} \|q_+\|_{L^1(\mathbb{R}_+)}^2. \qquad \square$$

Remark 3.3. *In the case of a truncated potential $\tilde{q}$, since $\widetilde{\gamma}_\pm \leq \gamma_\pm$ and $C(h, \tilde{q}_\pm) \leq C(h, q_\pm)$, all above results remain true for the same h. If, in addition, $q_+ \in L^2_{loc}(\mathbb{R}_+)$ then $\tilde{q}_+ \in \ell^\infty(L^2(\mathbb{R}_+))$ and thus $e^{-2hx}\widetilde{A}_+(x) \in L^2(\mathbb{R}_+)$ for h large enough.*

Corollary 3.4. *Let $q \in \ell^\infty(L^2(\mathbb{R}_-))$, and let $h \geq \gamma_-$ where γ_- is defined as in Proposition 3.2. Then*

(1) *$ik - m_-(k^2) \in L^2(\mathbb{R} + ih)$,*

(2) *if $\tilde{q}_- = q_-|_{[-a,0]}$ for some $a > 0$, then*

$$\delta m_-(k^2) := m_-(k^2) - \widetilde{m}_-(k^2) \to 0 \text{ in } L^2(\mathbb{R} + ih) \quad , \quad a \to \infty.$$

Proof. Note that for $k = \alpha + ih$ where $\alpha \in \mathbb{R}, h > \gamma_-$,

$$ik - m_-(k^2) = \int_{-\infty}^{0} e^{-2i\alpha x} e^{2hx} A_-(x) dx$$

where by Proposition 3.2, $e^{2hx}A_-(x) \in L^2(\mathbb{R} + ih)$. The Plancherel formula in our case takes the form:

$$\left\| \int_{\mathbb{R}_-} e^{-2i\alpha x} f(x) dx \right\|_{L^2(\mathbb{R})} = \sqrt{\pi} \, \| f(x) \|_{L^2(\mathbb{R}_-)}$$

and hence

$$\left\| ik - m_-(k^2) \right\|_{L^2(\mathbb{R}+ih)} = \sqrt{\pi} \left\| e^{2hx} A_-(x) \right\|_{L^2(\mathbb{R}_-)} < \infty$$

In the case $\tilde{q}_- = q_-|_{[-a,0]}$, we have $\widetilde{\gamma}_- \leq \gamma_-$ and by Proposition 3.2 (2),

$$\delta m_-(k^2) = \int_{-\infty}^{-a} e^{-2ikx} \delta A(x) dx$$

and therefore for any $h > \gamma_-$,

$$\left\| \delta m_-(k^2) \right\|_{L^2(\mathbb{R}+ih)} = \sqrt{\pi} \left\| e^{2hx} \delta A(x) \right\|_{L^2((-\infty,-a])} \to 0 \quad , \quad a \to \infty.$$

□

Chapter 4

Properties of the transmission and reflection coefficients

In this chapter, we establish some properties of one of our main objects:

$$G(k) := \Delta R(k) = R(k) - R_+(k).$$

As mentioned when first introduced, neither R nor R_+ can be analytically extended to the upper half plane for a potential q under the conditions of Hypothesis 2.2 (or those in Proposition 3.2). But by rewriting G exclusively in terms of R_-, T_+, and L_+, we will see that G can be analytically extended to the upper half plane. We also derive key properties of R_-, T_+, L_+ and G in $\mathbb{C}^+$ which will be used later to recover $q_-(x)$ assuming R, R_+ are known.

First we rewrite the reflection and transmission coefficients in terms of the m-function. Setting

$$m_\pm = m_\pm(k^2 + i0) := \lim_{\varepsilon \to 0^+} m_\pm(k^2 + i\epsilon),$$

we have:

$$R(k) = -\frac{m_- + \overline{m}_+}{m_- + m_+} \frac{\overline{\Psi_+(0,k)}}{\Psi_+(0,k)}, \tag{4.1}$$

$$R_+(k) = -\frac{\overline{m}_+ + ik}{m_+ + ik} \frac{\overline{\Psi_+(0,k)}}{\Psi_+(0,k)}, \tag{4.2}$$

$$R_-(k) = \frac{ik - m_-}{ik + m_-}, \tag{4.3}$$

$$L_+(k) = \frac{ik - m_+}{ik + m_+}, \tag{4.4}$$

$$T_+(k) = \frac{2ik}{(ik + m_+)\Psi_+(0,k)}. \tag{4.5}$$

The above are obtained using continuity of the various solutions and their derivatives in x at the point $x = 0$ or, alternately, the Wronskians. Recall that the above are defined for a.e. real k. From (4.3)-(4.4) and properties of the Titchmarsh-Weyl m-function, one notes that L_+, R_- have a meromorphic extension to the upper half plane. By Proposition 3.2 (3), L_+, R_- are smooth on $\mathbb{R} + ih$ for any $h > \gamma$. Furthermore, by the Borg-Marchenko uniqueness theorem L_+, R_- determine uniquely respectively $q_\pm(x)$ [11].

Now, using (2.3) and (4.1)-(4.5), one readily verifies that:

$$G(k) = \frac{T_+^2(k)R_-(k)}{1 - L_+(k)R_-(k)} \tag{4.6}$$

and thus G can be analytically extended to the upper half plane (recall that T_+ also has an analytic extension since q_+ is Faddeev [5]).

Proposition 4.1. *Let q be as in Proposition 3.2 and let*

$$h_\pm = \inf\left\{h \;:\; h > \gamma_\pm \textit{ and } C(h, q_\pm) < \frac{h}{2}\right\}$$

Then for all $h > \max(h_+, h_-)$

(1) *$R_-, L_+ \in \left(L^\infty \cap L^2\right)(\mathbb{R} + ih)$ with their L^∞ norm no greater than $1/3$.*

(2) *$\delta L_+ \;\to\; 0$ in $L^\infty(\mathbb{R} + ih)$ when $a \to \infty$.*

(3) *$R_- \in L^1(\mathbb{R} + ih)$.*

(4) *$\delta R_- \;\to\; 0$ in $L^1(\mathbb{R} + ih)$.*

Proof. The above results are direct consequences of Proposition 3.2. For all $k \in \mathbb{R} + ih$, such that $h > \max(h_+, h_-)$,

$$\begin{aligned}
\left|ik - m_\pm(k^2)\right| &= \left|\int_0^{\pm\infty} e^{\pm 2ikx} A_\pm(x)dx\right| \\
&\leq \left\|e^{\mp 2hx} A_\pm(x)\right\|_{L^1(\mathbb{R}_\pm)} \leq \frac{h}{2} \\
\left|ik + m_\pm(k^2)\right| &= \left|2ik \mp \int_0^{\pm\infty} e^{2ikx} A_\pm(x)dx\right| \\
&\geq 2\left|k\right| \cdot \left|1 - \frac{1}{2\left|k\right|}\left|\int_0^{\pm\infty} e^{\pm 2ikx} A_\pm(x)dx\right|\right| \\
&\geq \frac{3\left|k\right|}{2}
\end{aligned}$$

where we have used $|k| \geq h$. Thus $\frac{1}{ik + m_\pm(k^2)} \in L^2(\mathbb{R} + ih)$ and it follows immediately from (4.3)-(4.4) that $\|R_-(k)\|_{L^\infty(\mathbb{R}+ih)}, \|L_+(k)\|_{L^\infty(\mathbb{R}+ih)} \leq \frac{1}{3}$ and $L_+, R_- \in L^2(\mathbb{R} + ih)$.

We further obtain $R_- \in L^1(\mathbb{R}+ih)$ by the Cauchy-Schwartz inequality for $h > h_-$:

$$\|R_-(k)\|_{L^1(\mathbb{R}+ih)} \leq \left\| \frac{1}{ik + m_-(k^2)} \right\|_{L^2(\mathbb{R}+ih)} \cdot \left\| ik - m_-(k^2) \right\|_{L^2(\mathbb{R}+ih)}.$$

By Remark 3.3, the above is also true for $\widetilde{R}_-, \widetilde{L}_+$.

Now for all $k \in \mathbb{R}+ih$ with $h > h_+$,

$$\begin{aligned} |\delta L_+(k)| &= \left| \frac{-2ik\delta m_+(k^2)}{(ik + m_+(k^2))(ik + \widetilde{m}_+(k^2))} \right| \leq \frac{8}{9h} \left| \delta m_+(k^2) \right| \\ &\leq \frac{8}{9h} \left\| e^{-2hx} \delta A_+(x) \right\|_{L^1([a,\infty))} \to 0 \quad , \quad a \to \infty. \end{aligned}$$

We also have

$$\delta R_-(k) = \frac{-2ik\delta m_-(k^2)}{(ik + m_-(k^2))(ik + \widetilde{m}_-(k^2))}.$$

Using L^∞ norms and the Cauchy-Schwartz inequality:

$$\begin{aligned} \|\delta R_-\|_{L^1(\mathbb{R}+ih)} &\leq \left\| \frac{-2ik}{ik + \widetilde{m}_-(k^2)} \right\|_{L^\infty(\mathbb{R}+ih)} \cdot \left\| \frac{\delta m_-(k^2)}{ik + m_-(k^2)} \right\|_{L^1(\mathbb{R}+ih)} \\ &\leq \frac{4}{3} \left\| \frac{1}{ik + m_-(k^2)} \right\|_{L^2(\mathbb{R}+ih)} \cdot \left\| \delta m_-(k^2) \right\|_{L^2(\mathbb{R}+ih)} \end{aligned}$$

and the right hand side, by Corollary 3.4, goes to zero when $a \to \infty$. □

Remark 4.2. *Property* (3) *will play a crucial role. Note that if $q(x) = c\delta(x)$, where δ is Dirac's δ-function, then*

$$R(k) = \frac{c}{2ik - c}$$

which is not in $L^1(\mathbb{R}+ih)$. This suggests that the condition $\ell^\infty(L^2(\mathbb{R}))$ may not be relaxed to read $\ell^\infty(L^1(\mathbb{R}))$.

Corollary 4.3. *For any finite z, and q_-, h under the conditions of Proposition 4.1,*

$$e^{2ikz} R_-(k) \in L^1(\mathbb{R}+ih).$$

Proof. Immediately follows from $\left\| e^{2ikz} R_-(k) \right\|_{L^1(\mathbb{R}+ih)} = e^{-2hz} \left\| R_-(k) \right\|_{L^1(\mathbb{R}+ih)}$. □

While trivial, the above corollary plays an important part in our arguments. This stems from the fact that the reflection coefficient for the shifted potential $q(x+z)$ is $R(k)e^{2ikz}$ where $R(k)$ is the reflection coefficient corresponding to $q(x)$.

Lemma 4.4. *Let $q_+ \in L^1_1(\mathbb{R}_+)$ and let $h > \beta$ where*

$$\beta = 2\max\left\{\|q_+\|_{L^1_1(\mathbb{R}_+)}, 1\right\}.$$

Then

$$\|T_+(k)\|_{L^\infty(\mathbb{R}+ih)}, \left\|\widetilde{T}_+(k)\right\|_{L^\infty(\mathbb{R}+ih)} \le 2^\beta$$

and $\delta T_+(k) \to 0$ in $L^\infty(\mathbb{R}+ih)$ as $a \to \infty$.

Proof. The following are well-known facts (e.g. [5]) for q_+ Faddeev and supported on $\mathbb{R}_+$:

(1) $T_+(k)$ is analytic in $\mathbb{C}^+$ except at a finite number of simple poles $\{i\varkappa_n\}_{n=1}^N$ where

$$N \le 1 + \int_{\mathbb{R}_+} |x|\,|q_+(x)|\,dx. \tag{4.7}$$

(2) $|T_+(k)|^2 + |L_+(k)|^2 = 1$ for a.e. real k and $T_+(-k) = \overline{T_+(k)}$.

(3) $T_+(k)$ admits the following representation for any $k \in \mathbb{C}^+$:

$$T_+(k) = \prod_{n=1}^N \frac{k+i\varkappa_n}{k-i\varkappa_n}\exp\left(\frac{i}{\pi}\int_{\mathbb{R}} \frac{\log|T_+(\omega)|^{-1}}{\omega-k}d\omega\right).$$

We also have the Lieb-Thirring inequality [19]

$$\sum_{n=1}^N \varkappa_n \le L_{1/2,1}\int_{\mathbb{R}_+} |q_+(x)|\,dx \tag{4.8}$$

where $1/2 \le L_{1/2,1} \le 1.005$. Thus, for any $k \in \mathbb{R}+ih$, $h > \beta$

$$\begin{aligned}|T_+(k)| &= \left|\prod_{n=1}^N \frac{k+i\varkappa_n}{k-i\varkappa_n}\right| \exp\left(\frac{1}{\pi}\int_{\mathbb{R}} \operatorname{Re}\frac{i}{\omega-k}\log|T_+(\omega)|^{-1}d\omega\right)\\ &= \prod_{n=1}^N \sqrt{1+\frac{4h\varkappa_n}{(h-\varkappa_n)^2}}\exp\left(\frac{-h}{\pi}\int_{\mathbb{R}} \frac{\log|T_+(\omega)|^{-1}}{h^2+(\omega-\alpha)^2}d\omega\right).\end{aligned}$$

Since $|T_+(\omega)| \le 1$ for a.e. real ω, the above becomes

$$|T_+(k)| \le \prod_{n=1}^{N} \sqrt{1 + \frac{4h\varkappa_n}{(h-\varkappa_n)^2}}.$$

But by (4.8), we also have for each n:

$$\varkappa_n \le L_{1/2,1} \|q_+\|_{L^1(\mathbb{R}_+)} \le \frac{2}{3}\beta < \frac{2}{3}h \tag{4.9}$$

and hence for all $k \in \mathbb{R} + ih$ where $h > \beta$

$$|T_+(k)| \le \left(\frac{11}{3}\right)^{\beta/2} \le 2^{\beta} \tag{4.10}$$

where we have used $N \le \beta$ from (4.7). Inequalities (4.7)-(4.10) are also valid for $\widetilde{T}_+(k)$ and thus $T_+, \widetilde{T}_+$ are uniformly bounded on $\mathbb{R} + ih$.

From [5], we now use the following results

$$\frac{1}{T_+(k)} = 1 - \frac{1}{2ik}\int_{\mathbb{R}} q_+(x) y_+(x,k)dx$$

where $y_+(x,k) := e^{-ikx}\varphi_{\ell,+}(x,k)$ satisfies for $\operatorname{Im} k \ge 0$

$$\begin{aligned} &y_+(x,k) = 1 + \int_x^\infty D_k(t-x) q_+(t) y_+(t,k)dt\ ,\ D_k(y) = \frac{e^{2iky}-1}{2ik}, \\ &|y_+(x,k)| \le K(\beta)(1+|x|), \end{aligned} \tag{4.11}$$

where K is a constant depending only on β. Note that

$$\delta T_+(k) = \frac{T_+(k)\tilde{T}_+(k)}{2ik}\left[\int_{\mathbb{R}} q_+(x)\delta y_+(x,k)dx + \int_{\mathbb{R}} \tilde{y}_+(x,k)\delta q_+(x)dx\right]. \tag{4.12}$$

From (4.11),

$$\delta y_+(x,k) = \int_x^\infty D_k(t-x)\tilde{y}_+(t,k)\delta q_+(t)dt + \int_x^\infty D_k(t-x) q_+(t)\delta y_+(t,k)dt \tag{4.13}$$

and since for $k \neq 0$, $|D_k(y)| \leq \frac{1}{|k|}$ for all $y \geq 0$, $\operatorname{Im} k \geq 0$,

$$\begin{aligned} |\delta y_+(x,k)| &\leq \frac{K(\beta)}{|k|} \|\delta q_+\|_{L^1_1(\mathbb{R}_+)} + \int_x^\infty \frac{|q_+(t)|}{|k|} |\delta y_+(t,k)| \, dt \\ &\leq \frac{K(\beta)}{|k|} e^{\frac{\beta}{2|k|}} \|\delta q_+\|_{L^1_1(\mathbb{R}_+)} \quad , \quad k \neq 0 \quad , \quad \operatorname{Im} k \geq 0 \end{aligned}$$

by iteration on the Volterra integral equation for $\frac{\delta y_+(x,k)k}{K(\beta)\|\delta q_+\|_{L^1_1(\mathbb{R})}}$ derived from (4.13). Hence for (4.12), we have

$$\|\delta T_+(k)\|_{L^\infty(\mathbb{R}+ih)} \leq 2^{2\beta} \frac{K(\beta)}{\beta} \|\delta q_+\|_{L^1_1(\mathbb{R}_+)}$$

where the right hand side goes to zero for $a \to \infty$ by the dominated convergence theorem.

□

Proposition 4.5. *Under Hypothesis 2.2,*

$$\Delta R(k) := R(k) - R_+(k)$$

is analytic in $\mathbb{C}^+$ *except on a set*

$$\mathcal{S} = \{i\varkappa_n\} \cup \sigma \subset i\mathbb{R}_+$$

where

$$\{-\varkappa_n^2\} = \operatorname{Spec}_d(-\partial_x^2 + q_+) \quad , \quad \sigma = \{\lambda : \lambda^2 \in \operatorname{Spec}(-\partial_x^2 + q) \cap \mathbb{R}_-\} .$$

Proof. By direct computation, we have

$$G(k) = T_+(k) \frac{ik - m_-(k^2)}{m_+(k^2) + m_-(k^2)} g(k) \tag{4.14}$$

where

$$g(k) = \frac{T_+(k)}{1 + L_+(k)} = \frac{1}{\Psi_+(0,k)}.$$

We gather the following facts:

(1) it is well-known that $T_+(k), L_+(k)$ are analytic in $\mathbb{C}^+ \setminus \{i\varkappa_n\}_{n=1}^N$ where $\{-\varkappa_n^2\}$ is the negative simple discrete spectrum of $-\partial_x^2 + q_+$. So we also have that $g(k)$ is

meromorphic in $\mathbb{C}^+$ but with poles different from those of T_+, L_+. Poles of $g(k)$ correspond to the poles of m_+, i.e. κ's such that $\Psi_+(0,\kappa) = 0$.

(2) recall that $m_\pm$ is analytic in $\mathbb{C}^+$ except for some singularities[1] $\kappa \in i\mathbb{R}_+$, hence so is the term $\frac{ik - m_-(k^2)}{(m_- + m_+)(k^2)}$. Note that $m_+(k^2) + m_-(k^2) = 0$ corresponds to

$$W(\psi_-, \psi_+) = 0.$$

But if these two solutions to $\partial_x^2 + q$ are linearly dependent, it follows that $\psi_\pm(x,k) \in L^2(\mathbb{R})$ and so $k \in \sigma$.

(3) we now consider singularities of $m_\pm$. Only m_- may contain a continuous part but then it would be preserved and present in σ. So we need only analyze isolated singularities. These correspond exactly to $\psi_\pm(0,\kappa) = 0$ with $\psi_\pm'(0,\kappa) \neq 0$. But if $\psi_-(0,\kappa) = 0$, we can assume $\psi_+(0,\kappa) \neq 0$ (otherwise $\kappa \in \sigma$) and so $m_+(\kappa^2)$ finite. Therefore, by (4.14),

$$G(\kappa) = -\frac{T(\kappa)}{\psi_+'(0,\kappa)}$$

is finite unless $\kappa \in \{i\varkappa_n\}$. Now if $\psi_+(0,\kappa) = 0$, we have by (4.5) that ($\kappa \neq 0$)

$$T_+(\kappa) = \frac{2i\kappa}{\psi_+'(0,\kappa)}$$

is finite and by (4.14) since we can assume $m_-(\kappa^2)$ is finite, then

$$G(\kappa) = T_+(\kappa)\frac{i\kappa + m_-(\kappa^2)}{\psi_+'(0,\kappa)}$$

is finite too.

Thus, we find that $G(k)$ is analytic in $\mathbb{C}^+ \setminus (\{i\varkappa_n\} \cup \sigma)$. □

Remark 4.6. *By Proposition 3.2, we have* $\gamma > \sup |\mathcal{S}|$ *and thus* $G(k)$ *is smooth on* $\mathbb{R} + ih$ *for any* $h > \gamma$. *Note that* σ *need not be finite, but it is bounded.*

[1]Singularities of m_+ are a finite number of poles since q_+ is Faddeev [11] whereas the set of singularities of m_-, while bounded by γ_-, need not be made of poles – it could be continuous.

Proposition 4.7. *Let q be a real function on $\mathbb{R}$ such that $q \in \ell^\infty(L^2(\mathbb{R}_-)) \cap L^1_1(\mathbb{R}_+)$ and let $h > h_0$ where $h_0 = \max(h_+, h_-, \beta)$ where $h_\pm$ are as in Proposition 4.1 and β as in Lemma 4.4. Then*

$$G(k) = \Delta R(k) \in L^1(\mathbb{R} + ih) \quad \textit{and} \quad \delta G(k) \to 0 \textit{ in } L^1(\mathbb{R} + ih).$$

Proof. Omitting the variable k for brevity, from (4.6), Proposition 4.1 and Lemma 4.4, we have:

$$\|G\|_{L^1(\mathbb{R}+ih)} \leq \frac{9}{8} \cdot 2^{2h_0} \cdot \|R_-\|_{L^1(\mathbb{R}+ih)} < \infty.$$

By direct computation,

$$\delta G = \frac{G_1 \delta L_+ + G_2 \delta T_+ + G_3 \delta R_-}{(1 - L_+ R_-)(1 - \widetilde{L}_+ \widetilde{R}_-)} \tag{4.15}$$

where

$$G_1 = T_+^2 R_- \widetilde{R}_- \quad , \quad G_2 = R_-(1 - L_+ \widetilde{R}_-)(T_+ + \widetilde{T}_+) \quad , \quad G_3 = \widetilde{T}_+^2.$$

By Proposition 4.1,

$$\left\| \frac{1}{(1 - L_+ R_-)(1 - \widetilde{L}_+ \widetilde{R}_-)} \right\|_{L^\infty(\mathbb{R}+ih)} \leq \left(\frac{9}{8}\right)^2$$

so it is enough to show that each term inside the brackets in (4.15) goes to zero in $L^1(\mathbb{R} + ih)$ as $a \to \infty$. Indeed

$$\begin{aligned}
\|G_1 \delta L_+\|_{L^1(\mathbb{R}+ih)} &\leq \frac{2^{2h_0}}{3} \cdot \|R_-\|_{L^1(\mathbb{R}+ih)} \cdot \|\delta L_+\|_{L^\infty(\mathbb{R}+ih)} \ \to \ 0, \\
\|G_2 \delta T_+\|_{L^1(\mathbb{R}+ih)} &\leq \frac{10}{9} \cdot 2^{2h_0} \, \|R_-\|_{L^1(\mathbb{R}+ih)} \cdot \|\delta T_+\|_{L^\infty(\mathbb{R}+ih)} \ \to \ 0, \\
\|G_3 \delta R_-\|_{L^1(\mathbb{R}+ih)} &\leq 2^{2h_0} \cdot \|\delta R_-\|_{L^1(\mathbb{R}+ih)} \ \to \ 0.
\end{aligned}$$

□

Corollary 4.8. *Let z be a fixed real parameter and let q, h be as in Proposition 4.7. Then $G_z(k) := e^{2ikz} \Delta R(k) \in L^1(\mathbb{R} + ih)$ and $\delta G_z(k) \to 0$ in $L^1(\mathbb{R} + ih)$.*

Proof. Note that $G_z(k)$ correspond to the shifted potential $q(x + z)$. For such potential, $R_-(k)$ becomes $R_-(k)e^{2ikz}$, $T_+(k)$ remains the same and $L_+(k)$ becomes $L(k)e^{-2ikz}$. So by Corollary 4.3 and Proposition 4.7, $G_z(k) \in L^1(\mathbb{R} + ih)$ and $\delta G_z(k) \to 0$ in $L^1(\mathbb{R} + ih)$. □

Chapter 5

Inverse scattering procedure and trace class integral operators

In this chapter, we investigate the classical inverse scattering procedure as derived by Faddeev-Marchenko for Faddeev potentials [7] and the role trace class operators play in recovering the potential through a determinant formula in the above inverse scattering procedure – but adapted to our more general setting.

5.1 Classical Faddeev-Marchenko inverse scattering

In this section, we review the Faddeev-Marchenko inverse scattering procedure which allows to recover a real-valued, Faddeev potential from spectral characteristics and Jost solutions to the one-dimensional Schrödinger equation (2.1). Literature is abundant on the subject, see e.g. [1, 5, 6, 7]. We take the example of q_+ with Jost solutions $\varphi_{\ell,+}(x,k)$, $\varphi_{r,+}(x,k)$ defined by (2.6)-(2.7).

Now dropping the $+$ subscript, we rewrite (2.7) for $x, k \in \mathbb{R}$:

$$\overline{\varphi_\ell(x,k)} - T(k)\varphi_r(x,k) + R(k)\varphi_\ell(x,k) = 0. \tag{5.1}$$

The *Faddeev functions*

$$y_\ell(x,k) = e^{-ikx}\varphi_\ell(x,k) \quad , \quad y_r(x,k) = e^{ikx}\varphi_r(x,k) \tag{5.2}$$

have Volterra integral representations (see also (4.11)) and by studying their iterations, one finds that y_ℓ, y_r (and hence φ_ℓ, φ_r) are analytic in $k \in \mathbb{C}^+$ and continuous in $\mathbb{C}^+ \cup \mathbb{R}$ for each $x \in \mathbb{R}$. The following asymptotics are also found for large $|k|$:

$$|y_\ell(x,k)| = 1 + O\left(\frac{1}{|k|}\right) \quad , \quad |T(k)y_r(x,k)| = 1 + O\left(\frac{1}{|k|}\right) \tag{5.3}$$

where $O(1/|k|)$ depends on x. From this, we define a Fourier integral representation for y_ℓ with respect to the variable k in the form:

$$y_\ell(x,k) \;= 1 + \int_0^\infty k_x(s)e^{iks}ds \tag{5.4}$$

where

$$k_x(s) := \frac{1}{2\pi}\int_{\mathbb{R}}(y_\ell(x,k)-1)e^{-iks}dk \tag{5.5}$$

is defined for all s but satisfies $k_x(s) = 0$ for $s < 0$ and $k_x(s) \in \mathbb{R}$ for $s \geq 0$ [6]. We also have that $k_x(s) \in L^2(\mathbb{R}_+)$.

Plugging (5.4) into the Volterra integral representation (4.11) and applying the Fourier operator $\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iky}dk$ for $y > 0$, we have

$$\begin{aligned}
k_x(y) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iky}\int_x^\infty D_k(t-x)q(t)dtdk \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iky}\int_x^\infty D_k(t-x)q(t)\int_0^\infty k_t(s)e^{iks}dsdtdk \\
&= \frac{1}{4}\int_x^\infty \left[\operatorname{sgn}(2t-2x-y)+1\right]q(t)dtdk \\
&\quad + \frac{1}{4}\int_0^\infty\int_x^\infty q(t)k_t(s)\left[\operatorname{sgn}(2t-2x+s-y)-\operatorname{sgn}(s-y)\right]\right]dsdtdk
\end{aligned}$$

where we have used:

$$\int_{\mathbb{R}}\frac{e^{iak}}{k}dk = i\pi\operatorname{sgn}(a) \quad , \quad a \neq 0.$$

We can now further simplify the integral equation for k_x to

$$k_x(y) = \frac{1}{2}\int_{\frac{2x+y}{2}}^\infty q(t)dt + \frac{1}{2}\int_0^y\int_{\frac{2x+y}{2}}^\infty q(z-s)k_{z-s}(s)dzds, \quad y > 0.$$

From the above formula, one finds

$$k_x(0+) = \frac{1}{2}\int_x^\infty q(t)dt \quad , \quad q(x) = -2\partial_x k_x(0+) \tag{5.6}$$

where $k_x(0+) = \lim_{y\to 0+} k_x(y)$.

So now the problem reduces to finding k_x. The idea is to apply a Fourier transformation to (5.1) or rather to its equivalent in terms of the Faddeev functions. In the course of deriving the final Marchenko equation (also referred to as the Gelfand-Levitan-Marchenko equation), we will need information about the discrete spectrum for which we choose to give an exposition now.

Because of the asymptotics $\varphi_\ell(x,k) = e^{ikx} + o(1)$ for $x \to \infty$ and $\varphi_r(x,k) = e^{-ikx}$, $x < 0$[1], for bound states in $L^2(\mathbb{R})$, we must have $k = i\varkappa_n$ and

$$\varphi_\ell(x, i\varkappa_n) = \gamma_n \varphi_r(x, i\varkappa_n) \tag{5.7}$$

for a nonzero constant γ_n, also called *dependency constant*. Because of the bounds on large $|k|$, we also have that the number of eigenvalues must be finite.

Since we then have

$$W(\varphi_r(x, i\varkappa_n), \varphi_\ell(x, i\varkappa_n)) = 0,$$

by (2.11) the set $\{i\varkappa_n\}_{n=1}^N$ coincide with poles of $T(k)$. We will need to calculate the residues of $T(k)$ at the poles, and before doing so, we introduce the last component needed in the scattering data to determine uniquely the potential: *norming constants*. They are associated with the bound states and defined by:

$$c_n = \|\varphi_\ell(x, i\varkappa_n)\|^{-1}_{L^2(\mathbb{R})}. \tag{5.8}$$

Now, setting $a(k) = 1/T(k)$, one can rewrite (2.11) as:

$$2ika(k) = W(\varphi_r(x,k), \varphi_\ell(x,k)). \tag{5.9}$$

We now consider the derivative of (5.9) with respect to k. Computing ($\dot{f} = \partial_k f$ and omitting the variables)

$$\begin{aligned}
\partial_x W(\varphi_r, \dot{\varphi}_\ell) &= \varphi_r \partial_x^2 \dot{\varphi}_\ell - \dot{\varphi}_\ell \partial_x^2 \varphi_r \\
&= \varphi_r \left((q - k^2)\dot{\varphi}_\ell + 2k\varphi_\ell \right) + \dot{\varphi}_\ell (k^2 - q)\varphi_r \\
&= 2k\varphi_r\varphi_\ell, \\
\partial_x W(\dot{\varphi}_r, \varphi_\ell) &= -2k\varphi_r\varphi_\ell,
\end{aligned}$$

one finds that

$$i\dot{a}(i\varkappa_n) = \int_{\mathbb{R}} \varphi_r(x, i\varkappa_n)\varphi_\ell(x, i\varkappa_n)dx = (\gamma_n c_n^2)^{-1} \neq 0.$$

[1] In the general Faddeev case, i.e. q does not vanish on the left half line, one has $\varphi_r(x,k) = e^{-ikx} + o(1)$, $x \to -\infty$ and the rest of the reasoning is unchanged.

In the last equation, we have used the fact that [7]

$$\varphi_\ell(x, i\varkappa_n), \varphi_r(x, i\varkappa_n) \in \mathbb{R} \quad , \quad x \in \mathbb{R}.$$

So we find that the poles of $T(k)$ are simple and

$$\mathrm{Res}(T(k), i\varkappa_n) = i\gamma_n c_n^2. \tag{5.10}$$

We are now ready to derive the Marchenko equation. First, we rewrite (5.1) in the form:

$$\begin{aligned} 0 &= \overline{y_\ell(x,k)} - 1 - (T(k)y_r(x,k) - 1) + R(k)y_\ell(x,k)e^{2ikx} \\ &=: \ Y_1(x,k) \ + \qquad Y_2(x,k) \qquad + \ Y_3(x,k) \end{aligned} \tag{5.11}$$

Now we apply to (5.11) the Fourier operator

$$\mathcal{F}f(x,y) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iky} f(x,k)dk \quad , \quad y > 0.$$

We consider each term Y_i, $i = 1, 2, 3$ separately. For Y_1, by (5.4) and (5.5),

$$\begin{aligned} Y_1(x,k) &= \int_0^\infty \overline{k_x(s)} e^{-iks} ds \\ \mathcal{F}Y_1(x,y) &= \overline{k_x(y)} = k_x(y) \end{aligned} \tag{5.12}$$

since k_x is real for $y > 0$.

For Y_2, note that $Ty_r - 1$ is analytic in the upper half plane except at the poles $\{i\varkappa_n\}_{n=1}^N$. By closing the contour as in Figure 5.1, and taking into account the asymptotics (5.3), we have that Jordan's lemma applies and by the residue theorem,

$$\begin{aligned} \mathcal{F}Y_2(x,y) &= -\frac{1}{2\pi}\int_{\mathbb{R}} (T(k)y_r(x,k) - 1)\, e^{iky} dk \\ &= -i\sum_{n=1}^N \mathrm{Res}\left((T(k)y_r(x,k) - 1)\, e^{iky}, i\varkappa_n \right) \\ &= -i\sum_{n=1}^N \mathrm{Res}\,(T(k), i\varkappa_n) \cdot y_r(x, i\varkappa_n) \cdot e^{-\varkappa_n y} \end{aligned}$$

and by (5.2), (5.7), and (5.10), the above continues as

$$\begin{aligned}\mathcal{F}Y_2(x,y) &= -i\sum_{n=1}^{N} ic_n^2\gamma_n \cdot \frac{y_\ell(x,i\varkappa_n)}{\gamma_n}e^{-2\varkappa_n x}\cdot e^{-\varkappa_n y}\\ &= \sum_{n=1}^{N} c_n^2 e^{-\varkappa_n(2x+y)}\left(1+\int_0^\infty k_x(z)e^{-\varkappa_n z}dz\right)\\ &= \sum_{n=1}^{N} c_n^2 e^{-\varkappa_n(2x+y)} + \int_0^\infty k_x(z)\left(\sum_{n=1}^{N} c_n^2 e^{-\varkappa_n(2x+y+z)}\right)dz. \end{aligned} \tag{5.13}$$

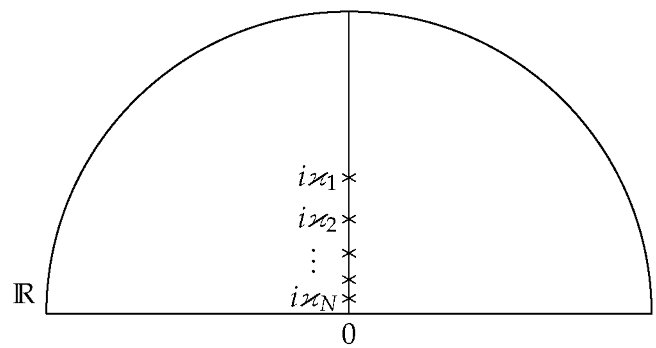

Figure 5.1. Contour of integration used to derive the Marchenko equation

For Y_3, we simply plug in (5.4) to find

$$\begin{aligned}\mathcal{F}Y_3(x,y) &= \frac{1}{2\pi}R(k)\left(1+\int_0^\infty k_x(z)e^{ikz}dz\right)e^{ik(2x+y)}dk\\ &= \frac{1}{2\pi}R(k)e^{ik(2x+y)}dk + \frac{1}{2\pi}\int_0^\infty k_x(z)\int_{\mathbb{R}} R(k)e^{ik(2x+y+z)}dkdz \end{aligned} \tag{5.14}$$

Defining the Marchenko kernel by:

$$M_x(\cdot) = M(\cdot + 2x), \tag{5.15}$$

$$M(s) = \sum_{n=1}^{N} c_n^2 e^{-\varkappa_n s} + \frac{1}{2\pi}\int_{\mathbb{R}} e^{iks}R(k)dk, \tag{5.16}$$

we regroup like terms in (5.12)-(5.14) so that the Fourier transform of (5.11) becomes the

Marchenko equation:

$$k_x(y) + M_x(y) + \int_0^\infty M_x(y+z)k_x(z)dz = 0 \quad , \quad y > 0. \tag{5.17}$$

We summarize the procedure as follows: if $q \in L^1_1(\mathbb{R})$, the scattering data consisting of

- the discrete spectrum $\{-\varkappa_n^2\}_{n=1}^N$ of the Schrödinger operator $-\partial_x^2 + q(x)$ on $L^2(\mathbb{R})$,
- norming constants $\{c_n\}_{n=1}^N$ as defined in (5.8), that is associated to the norm of the eigenfunctions of the Schrödinger operator,
- and the reflection coefficient $R(k)$, $k \in \mathbb{R}$

determine together the potential uniquely. From the above scattering data, one constructs the Marchenko operator as defined in (5.15)-(5.16), then solve the Marchenko equation (5.17) for k_x, and one recovers the potential through (5.6).

5.2 The Marchenko kernel for a potential supported on $\mathbb{R}_-$

We now stray from the case of q_+ to the case of a Faddeev potential supported on $\mathbb{R}_-$. Then the Marchenko kernel (5.16) can be rewritten in a more compact form which will prove useful in deriving its properties.

Indeed, if the Faddeev potential $q(x)$ is supported on $\mathbb{R}_-$, it is well-known (see e.g. [1]) that $R(k)$ can be analytically continued into $\mathbb{C}^+$, its poles are $\{i\varkappa_n\}_{n=1}^N$ and

$$\operatorname{Res}(R(k), i\varkappa_n) = ic_n^2, \tag{5.18}$$

where c_n is the norming constant associated to $\varkappa_n$ in the scattering data. This stems from the fact that φ_ℓ is now entire, and rewriting (5.1) as

$$\varphi_r(x,k) = \frac{1}{T(k)}\overline{\varphi_\ell(x,k)} + \frac{R(k)}{T(k)}\varphi_\ell(x,k)$$

we have that

$$\frac{R(i\varkappa_n)}{T(i\varkappa_n)} = \frac{1}{\gamma_n} \neq 0$$

and (5.18) follows from (5.10).

So for any $h > \max\{\varkappa_n\}$, one can deform the contour [12] in (5.16) and by the residue theorem rewrite the Marchenko kernel (5.16)

$$M(s) = \frac{1}{2\pi} \int_{\mathbb{R}+ih} e^{iks} R(k) dk. \tag{5.19}$$

Note that (5.19) can then be used for any compactly supported q's using a shifting argument.

5.3 A trace class operator

If we define the Marchenko operator on $L^2(\mathbb{R}_+)$ as:

$$(\mathbb{M}_x f)(y) = \int_0^\infty M_x(y+s) f(s) ds \quad , \quad f \in L^2(\mathbb{R}_+) \tag{5.20}$$

then (5.17) becomes

$$(1 + \mathbb{M}_x) k_x(y) = -M_x(y)$$

where 1 represents the identity operator. It is also known that $1 + \mathbb{M}_x$ is boundedly invertible [5].

Assuming the Fredholm determinant in (5.20) is well-defined, one can also rewrite (5.6) as [7]

$$q(x) = -2\partial_x^2 \log \det(1 + \mathbb{M}_x), \tag{5.21}$$

known as the Bargmann, Dyson, or determinant formula (see e.g. [12]). A formal derivation of (5.21) can be found in [7] with rigorous proofs only in the cases of reflectionless potentials or rational reflection coefficients. These cases correspond to degenerate kernels and hence representations with finite dimensional matrices. In the general case, we can ascertain the fact that the determinant is well-defined if $\mathbb{M}_x$ is trace class since [15]

$$\det(1 + A) \leq e^{\|A\|_{\mathfrak{S}_1}}.$$

However, we don't know if it is always the case for a generic Faddeev potential. In this section we introduce a lemma which appeared in a more general form in [13] and will be a central argument in the main result of this paper in the next chapter.

Proposition 5.1. *Let A be smooth on $\mathbb{R} + ih$ for some $h > 0$, and $A \in L^1(\mathbb{R} + ih)$. Then the integral operator $\mathbb{A}$ on $L^2(\mathbb{R}_+)$ with kernel*

$$\mathbf{A}(x,y) = \int_{\mathbb{R}+ih} e^{ik(x+y)} A(k) \frac{dk}{2\pi} \quad , \quad x,y \geq 0$$

is trace class, and

$$\|\mathbb{A}\|_{\mathfrak{S}_1} \leq \frac{1}{4\pi h} \|A\|_{L^1(\mathbb{R}+ih)} .$$

Proof. Denote

$$A_h(\alpha) := A(\alpha + ih) \quad , \quad \widehat{f}(z) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikz} f(k) dk.$$

Then rewrite $\mathbb{A}$ as an operator on $L^2(\mathbb{R})$ by considering $x, y \in \mathbb{R}$ and:

$$\mathbf{A}(x,y) = \chi(x) e^{-h(x+y)} \widehat{A_h}(x+y) \chi(y)$$

where χ is the characteristic function on $\mathbb{R}_+$. By convolution and a change of variable, we have:

$$\begin{aligned}
\widehat{A_h}(x+y) &= \left(\widehat{\sqrt{A_h}} * \widehat{\sqrt{A_h}} \right)(x+y) \\
&= \int_{\mathbb{R}} \widehat{\sqrt{A_h}}(x-s) \widehat{\sqrt{A_h}}(y+s) ds.
\end{aligned}$$

So $\mathbb{A} = \mathbb{A}_1 \mathbb{A}_2$ where $\mathbb{A}_1, \mathbb{A}_2$ are operators on $L^2(\mathbb{R})$ with kernels

$$\begin{aligned}
\mathbf{A}_1(x,s) &= \chi(x) e^{-hx} \widehat{\sqrt{A_h}}(x-s), \\
\mathbf{A}_2(s,y) &= \chi(y) e^{-hy} \widehat{\sqrt{A_h}}(y+s).
\end{aligned}$$

One readily has

$$\begin{aligned}
\|\mathbb{A}_k\|_{\mathfrak{S}_2}^2 &= \iint_{\mathbb{R}^2} |\mathbf{A}_k(\xi,\eta)|^2 d\xi d\eta \\
&= \int_{\mathbb{R}} \chi(z) e^{-2hz} dz \int_{\mathbb{R}} \left| \widehat{\sqrt{A_h}}(S) \right|^2 dS \\
&= \frac{1}{4\pi h} \left\| \sqrt{A_h} \right\|_{L^2(\mathbb{R})}^2 = \frac{1}{4\pi h} \|A_h\|_{L^1(\mathbb{R})}
\end{aligned}$$

where we have used the Plancherel equality

$$\left\|\widehat{f}\right\|_2^2 = \frac{1}{2\pi}\left\|f\right\|_2^2$$

and hence

$$\|\mathbb{A}\|_{\mathfrak{S}_1} \leq \|\mathbb{A}_1\|_{\mathfrak{S}_2}\|\mathbb{A}_2\|_{\mathfrak{S}_2} = \frac{1}{4\pi h}\|A_h\|_{L^1(\mathbb{R})} = \frac{1}{4\pi h}\|A\|_{L^1(\mathbb{R}+ih)}. \qquad \square$$

Chapter 6

The determinant formula

We now present our main result which gives a formula to recover a nondecaying unknown potential q_- assuming that q_+ and the reflection coefficient R are known.

6.1 The main result

We will consider a potential which is locally square integrable on the line, in $\ell^\infty(L^2(\mathbb{R}_-))$ and such that q_+ is Faddeev class. The classical inverse scattering results do not apply directly since q is not Faddeev and the negative part of the spectrum of $-\partial_x^2 + q(x)$ need not be finite. However, since $q \in L^2_{loc}(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$, we have

- $\tilde{q} \in L^1_1(\mathbb{R})$ so the classical inverse scattering procedure applies to the truncated potential,
- $\tilde{q}$ is compactly supported so we can use (5.19),
- $\tilde{q} \in \ell^\infty(L^2(\mathbb{R}))$ which we will show implies that $\widetilde{\mathbb{M}}_x$ is trace class and so (5.21) applies.

The above will be a basis for our limiting procedure.

Theorem 6.1. *Let q be a real, locally square integrable potential on $\mathbb{R}$ such that ($q_\pm = q|_{\mathbb{R}_\pm}$)*

- $\sup\limits_{x \le 0} \displaystyle\int_{x-1}^{x} |q_-(s)|^2\, ds < \infty$ *(uniformly in L^2_{loc}),*
- $\displaystyle\int_{\mathbb{R}_+} (1+x)\, |q_+(x)|\, dx < \infty$ *(Faddeev)*

and let $R(k)$, $R_+(k)$ be the right reflection coefficient corresponding to q, q_+ respectively.

Let $\mathbb{M}_x^+$ be the Marchenko operator associated with the scattering data

$$\left\{R_+(k), -\varkappa_n^2, c_n\right\}_{k \in \mathbb{R}, 1 \le n \le N}$$

for q_+ (given by (5.20) *and* (5.15)-(5.16)*) and let $\mathbb{G}_x$ be the following integral operator associated* *with $R - R_+$:*

$$(\mathbb{G}_x f)(y) = \int_0^\infty \mathbf{G}_x(y+s)f(s)ds \quad , \quad f \in L^2(\mathbb{R}_+), \tag{6.1}$$

$$\mathbf{G}_x(s) = \frac{1}{2\pi}\int_{\mathbb{R}+ih} e^{ik(s+2x)}(R - R_+)(k)dk \tag{6.2}$$

with some $h > 0$ sufficiently large.

Then for any $x < 0$

$$q_-(x) = -2\partial_x^2 \log\det\left(1 + \left(1 + \mathbb{M}_x^+\right)^{-1}\mathbb{G}_x\right) \tag{6.3}$$

with the determinant defined in the classical Fredholm sense.

Proof. We will first prove the statement for $\tilde{q}$. For a fixed $a > 0$, $\tilde{q}$ is compactly supported. Hence, $\widetilde{R}$ can be analytically continued in $\mathbb{C}^+$ except at a finite number of poles $\{i\widetilde{\varkappa}_n\}_{n=1}^{\widetilde{N}}$, and the Marchenko kernel (5.16) becomes

$$\widetilde{M}(s) = \frac{1}{2\pi}\int_{\mathbb{R}+ih} e^{iks}\widetilde{R}(k)dk \quad , \quad h > \max\{\widetilde{\varkappa}_n\}_{n=1}^{\widetilde{N}}.$$

Define $q_a(x) = \tilde{q}(x+a)$ as in Figure 6.1. Then q_a is supported on $\mathbb{R}_-$ and $q_a \in \ell^\infty(L^2(\mathbb{R}_-))$.

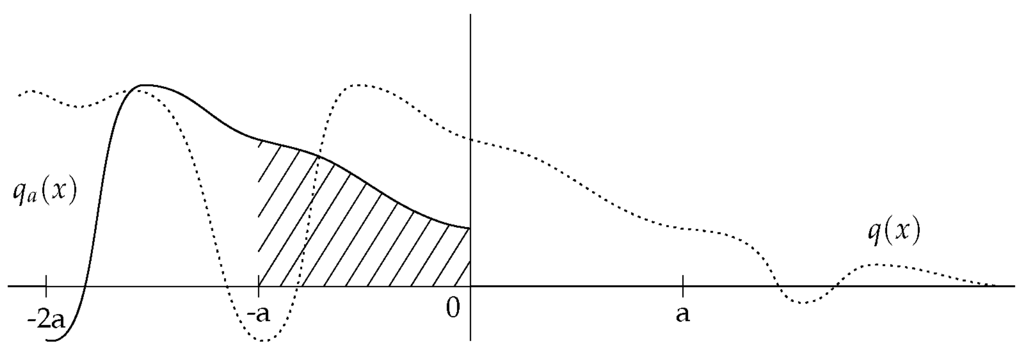

Figure 6.1. Shifted potential $q_a(x) = \tilde{q}(x+a)$

Now its reflection coefficient $R_{a,-}(k) \in L^1(\mathbb{R} + ih)$ for $h > h_a$ where h_a is defined as in Proposition 4.1 for q_a. But $\widetilde{R}(k) = R_{a,-}(k)e^{-2ika}$, so they share the same poles and by Corollary 4.3, $\widetilde{R}(k), \widetilde{R}(k)e^{2ikx} \in L^1(\mathbb{R} + ih)$ for $h > h_a$. We also have that $\widetilde{R}(k)e^{2ikx}$ is smooth on $\mathbb{R} + ih$ for $h > \max\{\widetilde{\varkappa}_n\}_{n=1}^{\widetilde{N}}$ so by Proposition 5.1, $\widetilde{\mathbb{M}}_x$ is trace class. Hence the following

Bargmann formula applies:

$$\tilde{q}(x) = -2\partial_x^2 \log\det(1 + \widetilde{\mathbb{M}}_x). \tag{6.4}$$

Now write

$$\widetilde{R} = \widetilde{R}_+ + \widetilde{G} \quad , \quad \widetilde{G} = \Delta\widetilde{R} = \widetilde{R} - \widetilde{R}_+$$

and split the Marchenko operator accordingly[1]:

$$\widetilde{\mathbb{M}}_x = \widetilde{\mathbb{M}}_x^+ + \widetilde{\mathbb{G}}_x.$$

The same h_a is enough to ensure $\widetilde{\mathbb{M}}_x^+ \in \mathfrak{S}_1$ since $\widetilde{R}_+(k)e^{2ika}$ corresponds to $\tilde{q}_a(x)$ (above the shaded region in Figure 6.1).

In addition, for $h > h_0$ where h_0 is the same[2] as in Proposition 4.7,

$$\widetilde{G}, G, \delta G \in L^1(\mathbb{R} + ih)$$

and by Proposition 4.5 and the subsequent remark, we also have $\widetilde{G}, G, \delta G$ smooth on $\mathbb{R} + ih$. Since by Corollary 4.8, the same applies to $\widetilde{G}_x, G_x, \delta G_x$, we can apply Proposition 5.1 and conclude that $\widetilde{\mathbb{G}}_x, \mathbb{G}_x, \delta\mathbb{G}_x \in \mathfrak{S}_1$.

Therefore, first we rewrite the Bargmann formula (6.4) as:

$$\begin{aligned} \tilde{q}(x) &= -2\partial_x^2 \log\det(1 + \widetilde{\mathbb{M}}_x^+ + \widetilde{\mathbb{G}}_x) \\ &= -2\partial_x^2 \log\det(1 + \widetilde{\mathbb{M}}_x^+) - 2\partial_x^2 \log\det(1 + (1 + \widetilde{\mathbb{M}}_x^+)^{-1}\widetilde{\mathbb{G}}_x) \end{aligned} \tag{6.5}$$

where we have used the fact from classical Marchenko theory that $1 + \widetilde{\mathbb{M}}_x$ is boundedly invertible. But

$$\tilde{q}_+(x) = -2\partial_x^2 \log\det(1 + \widetilde{M}_x^+)$$

and $q_+(x) = 0$ for $x < 0$. So (6.5) becomes for $x < 0$:

$$\tilde{q}_-(x) = -2\partial_x^2 \log\det(1 + (1 + \widetilde{\mathbb{M}}_x^+)^{-1}\widetilde{\mathbb{G}}_x). \tag{6.6}$$

[1]Because $\tilde{q}_+$ is compactly supported, $\widetilde{M}^+$ can be equivalently expressed as (5.16) or (5.19).

[2]Note that h_0 is independent of a.

Now, we now use the fact that $(1+\mathbb{M}_x)$ remains boundedly invertible in its limit so the right hand side of (6.3) is well-defined. In addition, by Proposition 5.1, for $h > h_0$

$$\|\delta\mathbb{G}_x\|_{\mathfrak{S}_1} \leq \frac{1}{4\pi h}\|\delta G_x\|_{L^1(\mathbb{R}+ih)}$$

and the right hand hand side of the inequality goes to zero by Corollary 4.8 for $a \to \infty$. Therefore, we find indeed that the limit of (6.6) is (6.3). $\square$

6.2 Conclusions

Theorem 6.1 solves the inverse scattering problem for a steplike potential with the knowledge of its short range or Faddeev part. Indeed, given (Faddeev) q_+ one solves the direct scattering problem and find the scattering data

$$\{R_+(k), -\varkappa_n^2, c_n\}_{k\in\mathbb{R}, 1\leq n\leq N}$$

for q_+. Then, we construct $\mathbb{M}_x^+$ by (5.20) and (5.15)-(5.16) and given the (right) reflection coefficient for the whole potential q, one constructs by (6.1)-(6.2) the operator $\mathbb{G}_x$. The unknown (non decaying) part of q is recovered for each $x < 0$ by (6.3).

Note that if $\mathbb{M}_x^+ \in \mathfrak{S}_1$ then (6.3) simplifies to

$$q(x) = -2\partial_x^2 \log\det(1+\mathbb{M}_x), \quad x \in \mathbb{R}, \tag{6.7}$$

where $\mathbb{M}_x = \mathbb{M}_x^+ + \mathbb{G}_x$. It is of course well-known (see e.g. [5]) that under our condition on q_+, $\mathbb{M}_x \in \mathfrak{S}_2$ but we couldn't prove it for $\mathfrak{S}_1$. However, since $\mathbb{M}_x \in \mathfrak{S}_2$, then $\det(1+\mathbb{M}_x)$ can, in fact, be regularized differently from (6.3) (see [13] for details).

Further possible work also include finding how to deform the contour in (6.2) back to the real line – where reflection coefficients are measured in physical applications. Yet, we emphasize that the fact that (6.3) is understood in the classical sense is indeed quite important as it guarantees the convergence of various types of approximation of $(1+\mathbb{M}_x^+)^{-1}\mathbb{G}_x$ in trace norm. This, in turn, means a certain stability of the inverse problem algorithm based upon (6.3). Our result could have applications in particular in neutron reflectometry [2] and reflection seismology [4, 16]. In the first case, a material with unknown property is coated with a known material and neutron beams are sent with their reflection measured. In reflection seismology, seismic waves are sent to probe layers of the Earth. Higher energies are needed to penetrate deeper and their reflection will accurately depict the deeper layers if the algorithm is stable. Both cases may represent a partially known potential since the shallower layers may be assumed to be known through successive measurements.

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